Kelly Criterion

John Knight
The University of Texas Rio Grande Valley

MATH-6365 Probability & Statistics

Prof. Santanu Chakraborty

December 13, 2023

Introduction

John L. Kelly, Jr. was born in 1923 in Corsicana, Texas. After serving in the Naval Air Force during World War 2, he studied at the University of Texas in Austin and completed his PhD in Physics. This earned Kelly a job at the prestigious Bell Laboratories, owned and operated by AT&T.

At the time, AT&T had a virtual monopoly on American communications, which gave them plenty of resources to fund research. In fact, Bell Labs was described as being "like a university except its researchers didn't have to teach". This enabled Kelly to further his knowledge by collaborating with great mathematicians such as Claude Shannon, Ed Thorp, and John Van Neumann.

It was during his time at Bell Labs that Kelly authored the 1956 paper that would make him famous. The paper was entitled A New Interpretation of Information Rate and concerned the emerging field of information theory that had been developed by Shannon and others. However, the bulk of the paper primarily concerned gambling and probability theory – Kelly worried that this topic may prove too controversial in the conservative United States, which is why he chose the rather mundane title. It is within this paper that Kelly derived the formula that would make him famous – what is today known as the Kelly criterion.

The Hypothetical Problem

In his paper, Kelly suggested the following problem. Suppose a gambler has access to a private wire and is able to bet on the outcome of baseball games when he knows the result before the bookmaker. If the wire is always correct, it is obvious that the gambler should stake his entire bankroll on every bet, since he cannot lose. However, Kelly's question was, if the probability p of the wire being correct is less than 1 – in other words, sometimes the wire gives the incorrect winner – how much of his bankroll should the gambler risk?

The problem is interesting because of two conflicting notions. Obviously, the gambler wants to maximize his return from each bet. However, equally obviously, the gambler does not want to go bust. Let us denote the following:

f = the proportion of bankroll wagered

p = probability of a win

q = probability of a loss

b = odds given (for example 2 would represent 2-to-1)

w_0 = starting wealth

w_n = wealth after n bets

Then, the gambler's expected wealth after one bet can be calculated as follows:

$$E(w_1) = w_0 pbf - w_0 qf$$

$$= w_0 f(pb - q)$$

Note that w_0 is fixed, and that the term $pb - q$ will always be greater than 0 when the bet is advantageous to the gambler (in other words, the odds being offered are favorable compared to the true probability of winning). Therefore, we see that to maximize $E(w_1)$ the gambler needs to maximize f, which means staking 100% of his bankroll.

We can also see that if the gambler risks his entire bankroll on a single bet then he will go bust with probability q. When a gambler continues to risk his entire bankroll on a series of bets, the probability of going bust after n bets is $1 - p^n$, and it is easy to see that $\lim_{n \to \infty} 1 - p^n = 1$ when $p < 1$. So we have two conflicting facts: the gambler should stake his entire bankroll on each bet, but is certain to lose all his money in the long run by doing this!

Kelly's Solution

Kelly showed that the gambler can maximize the long term growth of his bankroll by staking $p - \frac{q}{b}$ on each bet. This formula came to be known as the Kelly criterion, and it maximizes the geometric mean of outcomes. As an example, let us suppose there is a biased coin with 55% chance of coming up heads, while an unsuspecting bookmaker is offering 1:1 odds. Then, with $p = 0.55$, $q = 0.45$, and $b = 1$, the recommended wager is $0.55 - \frac{0.45}{1} = 0.1$, or 10% of the gambler's bankroll.

A simple proof of the Kelly criterion follows. First, we can write the expected growth rate of the gambler's bankroll G as

$$G = (1 + fb)^p \cdot (1 - f)^q$$

To maximize the growth rate, we want to find the maximum of G as a function of f. This is easier if we first take the logarithm of both sides:

$$E = log(G) = p\ log(1 + fb) + q\ log(1 - f)$$

Now we need to differentiate E and find where it is equal to zero.

$$\frac{dE}{df} = \frac{pb}{1 + fb} + \frac{q}{1 - f} = 0$$

Rearranging, we have:

$$pb(1 - f) - q(1 + bf) = 0$$

$$pb - pbf - q - qbf = 0$$

$$pb - q - f(pb + qb) = 0$$

$$f = \frac{pb - q}{pb + qb}$$

$$f = \frac{pb - q}{b(p + q)}$$

And since $(p + q) = 1$,

$$f = \frac{pb - q}{b}$$

or

$$f = p - \frac{q}{b}$$

…which is the Kelly criterion.

Example

I wrote a simulation in R that used four competing strategies to bet on 500 tosses of a biased coin with 55% chance of coming up heads, with 1:1 odds offered on each bet. Each hypothetical gambler began with a $100 bankroll and the four strategies were as follows:

1. Kelly – staking the amount recommended by the Kelly criterion.
2. Double Kelly – a more aggressive strategy, staking double the amount recommended by the Kelly criterion.
3. Half Kelly – a more conservative strategy, staking half the amount recommended by the Kelly criterion.
4. Full Bank – the entire bankroll is risked on every bet.

The results of the simulation are shown in Figure 1. The Full Bank gambler quickly loses a bet and his bankroll drops to zero, where it obviously remains since he can no longer place any bets. The Double Kelly gambler actually starts well, and has a comfortable lead after around 40 tosses, but soon encounters sharp losses and drops to almost zero before recovering slightly. The Half Kelly gambler has the least dramatic trendline, with fewer big wins and losses, and in fact has the largest bankroll after as many as 400 bets. Eventually, however, the Kelly strategy wins out, and by the end of the simulation has a bankroll of around double its nearest competitor. It should be noted that this is not guaranteed, no matter how large the n; it is only as n approaches infinity that the Kelly criterion is sure to beat all other systems.

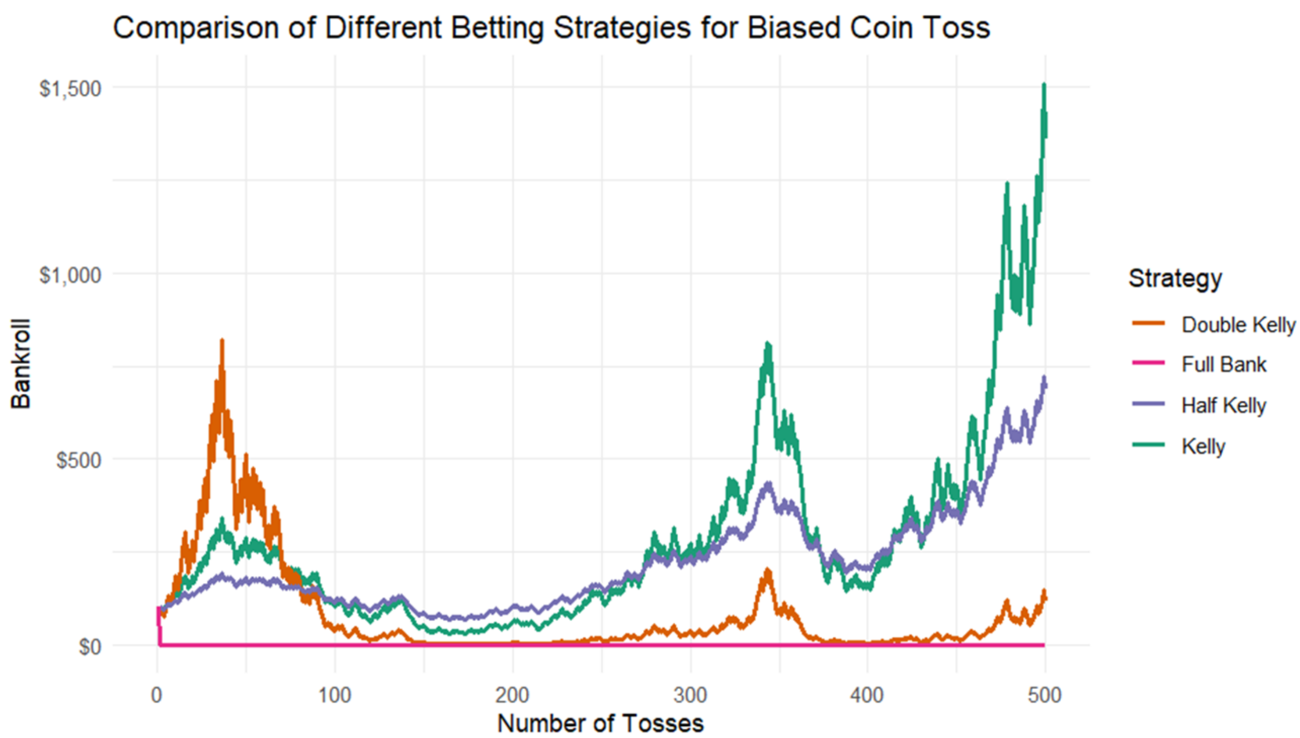

Figure 1. Simulation of four strategies to gamble on 500 coin tosses.

Daniel Bernoulli

In 1738, the famous mathematician Daniel Bernoulli wrote an article about a paradox known as the St. Petersburg wager. The wager is formulated as follows: Suppose you were to continue tossing a fair coin until heads appears. If heads appears on the first toss, you win \$2 (Bernoulli used ducats, but we shall stick with dollars here for simplicity). If heads appears on the second toss, you win \$4. On the third toss, \$8, and so on.

So the expected win from a single playing of this game is:

$$E = \sum_{i=1}^{\infty} \frac{1}{2i} \cdot 2i$$

$$= \infty$$

The paradoxical nature of this wager stems from the idea that in spite of the supposedly infinite expected returns, most people intuitively would not pay a large amount of money to play this game. The parallels with Kelly's paper occur because Bernoulli likewise suggested in his article that the optimal strategy would be to maximize the geometric mean.

In modern probability and statistics, the term ‘mean’ is widely used under the assumption it is referring to the arithmetic mean. That is,

$$A = \frac{1}{n}\sum_{i=1}^{n} a_i$$

However, a less common but nonetheless equally valid mean is its cousin the geometric mean:

$$G = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$$

According to Jensen’s inequality, the arithmetic mean is always greater than or equal to the geometric mean. This creates these paradoxical situations where optimizing the return of a single bet may involve a different strategy than maximizing the long-term growth of a bankroll, or of considering the utility function of wealth as it increases.

Kelly Criterion in Real Life

The Kelly criterion as stated in this report, that is, $p - \frac{q}{b}$, applies to solitary bets with a discrete win and loss amount. There are more advanced forms of the Kelly criterion that can be applied, for example, to stock market investments where wins and losses are variable, or to situations where many bets are placed simultaneously and the risk of ruin nonetheless needs to be avoided.

Although the Kelly system has been proven theoretically to maximize bankroll growth, it also has some downsides. Firstly, it naturally involves some very volatile swings due to the random nature of results, and gamblers are likely to experience some sharp reductions in wealth along the way. Also, in reality it is often impractical for gamblers to know the true value of p when they are placing a bet. Indeed, it is almost inevitable due to human nature than gamblers will tend to overestimate their edge. For this reason, it is often advised to use some fraction of the recommended Kelly stake, both to mitigate volatility and to adjust the value of p somewhat closer to the market consensus.

References

Bernoulli, D. (1738). Hydrodynamica. *Dulsecker. Consultable en ligne http://imgbase-scd-ulp. u-strasbg. fr/displayimage. php, 1738*.

Kelly, J. L. (1956). A new interpretation of information rate. *the bell system technical journal*, *35*(4), 917-926.

Poundstone, W. (2010). *Fortune's formula: The untold story of the scientific betting system that beat the casinos and Wall Street*. Hill and Wang.

Thorp, E. O. (2008). The Kelly criterion in blackjack sports betting, and the stock market. In *Handbook of asset and liability management* (pp. 385-428). North-Holland.